ACADEMIC PRESS

J. Math. Anal. Appl. 273 (2002) 667–676

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.academicpress.com

$A_r^\lambda(\Omega)$-weighted imbedding inequalities for A-harmonic tensors

Bing Liu

Department of Mathematical Sciences, Saginaw Valley State University, University Center, MI 48710, USA

Received 9 July 2001

Submitted by B. Mongiorno

Abstract

We prove new versions of $A_r^\lambda(\Omega)$-weighted imbedding inequalities for A-harmonic tensors locally and globally. The results are used to estimate the integrals of a homotopy operator T from the Banach space $L^s(\Omega,\wedge^l)$ to the Sobolev space $W^{1,s}(\Omega,\wedge^{l-1})$, $l=1,2,\ldots,n$, and Sobolev–Poincaré imbedding inequality with $A_r^\lambda(\Omega)$ weight.

1. Introduction

Imbedding inequalities for differential forms were proved by Iwaniec and Lutoborski, see [7]. Those inequalities are important for studying L^p theory of differential forms, continuum mechanics and quasiconformal mappings, see [1,2, 5–8]. We extend those inequalities, mainly the Theorems A and B, to new versions with $A_r^\lambda(\Omega)$ weight. The resulting inequalities are more general, hence can be used broadly. We first state the results from [7].

Theorem A. *Let D be a bounded, convex domain in $\mathbf{R}^n$. For each $s>1$ the integral* (1.9) *defines a bounded operator*

$$T : L^s(D,\wedge^l) \to W^{1,s}(D,\wedge^{l-1}) \tag{1.1}$$

E-mail address: bliu@svsu.edu.

0022-247X/02/$ – see front matter
PII: S0022-247X(02)00331-1

for $l = 1, 2, \ldots, n$ with norm estimated by

$$\|T\omega\|_{W^{1,s}(D)} \leqslant C(s,n)|D|\|\omega\|_{s,D}. \tag{1.2}$$

Theorem B. *Let $\omega \in L^s_{\mathrm{loc}}(D, \wedge^l)$, $d\omega \in L^s_{\mathrm{loc}}(D, \wedge^{l+1})$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be differential forms in a bounded domain $D \subset \mathbf{R}^n$. Then $d(T\omega)$ is a regular distribution of class $L^s_{\mathrm{loc}}(D, \wedge^l)$, and*

$$\omega = d(T\omega) + T(d\omega).$$

Moreover,

$$\|T\omega\|_{s,D} \leqslant C(n)|D| \operatorname{diam}(D)\|\omega\|_{s,D} \tag{1.3}$$

and

$$\big\|d(T\omega)\big\|_{s,D} \leqslant \|\omega\|_{s,D} + C(n)|D| \operatorname{diam}(D)\|d\omega\|_{s,D}, \tag{1.4}$$

and also

$$\big\|\nabla(T\omega)\big\|_{s,D} \leqslant C(n,s)|D|\|\omega\|_{s,D}, \tag{1.5}$$

where T is defined as in (1.9).

Now we need to introduce related concepts and definitions to the Theorems A and B. We shall adopt notations from [7,8].

We assume that Ω is a bounded connected open subset of $\mathbf{R}^n$. The Lebesgue measure of a set $D \subset \mathbf{R}^n$ is denoted by $|D|$. We denote balls as B and σB is the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. We call w a weight if $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $w > 0$ a.e. Let $e_1, e_2, \ldots, e_n$ be the standard orthonormal basis of $\mathbf{R}^n$. Assume that $\wedge^l = \wedge^l(\mathbf{R}^n)$ is the space of all l-forms in $\mathbf{R}^n$ spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$ corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$. An l-form $\omega(x) \in \wedge^l(\mathbf{R}^n)$, for $0 \leqslant l \leqslant n$ can be written as

$$\omega(x) = \sum \omega_{i_1,i_2,\ldots,i_l}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l},$$

where $\omega_{i_1,i_2,\ldots,i_l}(x)$ are real functions in $\mathbf{R}^n$, and $I = (i_1, i_2, \ldots, i_l)$, $i_j \in \{1, 2, \ldots, n\}$ and $j = 1, 2, \ldots, l$. Note that a differential 0-form is a differentiable function $f : \mathbf{R}^n \to \mathbf{R}$. The direct sum $\wedge = \bigoplus_{l=0}^n \wedge^l$, where $\wedge^0(\mathbf{R}^n) = \mathbf{R}$, is a grade algebra with respect to the wedge (exterior) product. The wedge product $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ belongs to $\wedge^{l+k}(\mathbf{R}^n)$ for $\alpha \in \wedge^l(\mathbf{R}^n)$ and $\beta \in \wedge^k(\mathbf{R}^n)$. The vector space $\wedge(\mathbf{R}^n)$ of all forms is equipped with the inner product $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$, where $\alpha = \sum \alpha^I e_I$ and $\beta = \sum \beta^I e_I$ are in $\wedge$. The Hodge star operator $* : \wedge \to \wedge$ is defined by $*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle (*1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by $|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \wedge^0 = \mathbf{R}$.

We use $D'(\Omega, \wedge^l)$ to denote the space of all differential l-forms. The exterior differential $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ is expressed as

$$d\omega(x) = \sum_{k=1}^{n} \sum_{I} \frac{\partial \omega_{i_1, i_2, \ldots, i_l}(x)}{\partial x_k}(x)\, dx_{i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l},$$

where $I = (i_1, \ldots, i_l)$.

The formal adjoint operator (Hodge codifferential) is given by

$$d^\star = (-1)^{nl+1} \star d \star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l).$$

$L^p(\Omega, \wedge^l)$ represents a space of differential l-forms with coefficients in $L^p(\Omega, \mathbf{R}^n)$ and it is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega \left|\omega(x)\right|^p dx \right)^{1/p} = \left(\int_\Omega \left(\sum_I \left|\omega_I(x)\right|^2 \right)^{p/2} dx \right)^{1/p}.$$

For $\omega \in D'(\Omega, \wedge^l)$ the vector-valued differential form

$$\nabla\omega = \left(\frac{\partial\omega}{\partial x_1}, \ldots, \frac{\partial\omega}{\partial x_n} \right)$$

consists of differential forms $\partial\omega/\partial x_i \in D'(\Omega, \wedge^l)$, where the partial differentiation is applied to the coefficients of ω.

The Sobolev space of l-forms is defined as $W^{1,p}(\Omega, \wedge^l) = L^p(\Omega, \wedge^l) \cap L_1^p(\Omega, \wedge^l)$ with norm

$$\|\omega\|_{W^{1,p}(\Omega,\wedge^l)} = (\operatorname{diam}\Omega)^{-1} \|\omega\|_{p,\Omega} + \|\nabla\omega\|_{p,\Omega}.$$

Similarly, the weighted norm of $w \in W^{1,p}(\Omega, \wedge^l)$ over Ω is defined by

$$\|\omega\|_{W^{1,p}(\Omega,\wedge^l),w^\alpha} = (\operatorname{diam}\Omega)^{-1} \|\omega\|_{p,\Omega,w^\alpha} + \|\nabla\omega\|_{p,\Omega,w^\alpha}, \tag{1.6}$$

where α is a real number and

$$\|\omega\|_{p,\Omega,w^\alpha} = \left(\int_\Omega \left|\omega(x)\right|^p w(x)^\alpha\, dx \right)^{1/p}.$$

An A-harmonic equation for differential forms is

$$d^\star A(x, d\omega) = 0, \tag{1.7}$$

where $A : \Omega \times \wedge^l(\mathbf{R}^n) \to \wedge^l(\mathbf{R}^n)$ satisfies the following conditions:

$$\left|A(x,\xi)\right| \leqslant a|\xi|^{p-1} \quad \text{and} \quad \left\langle A(x,\xi), \xi \right\rangle \geqslant |\xi|^p$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.7).

When considering real functions in $\mathbf{R}^n$, Eq. (1.7) is in the form $\operatorname{div} A(x, \nabla u) = 0$. There are many studies and well developed theory about it, for example, see [2,6].

A solution to (1.7) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi\rangle = 0$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support.

Definition 1.1. We call u an A-harmonic tensor in Ω if u satisfies the A-harmonic equation (1.7) in Ω.

If $\omega : \Omega \to \wedge^l$, then the value of $\omega(x)$ at the vectors $\xi_1, \xi_2, \dots, \xi_l \in \mathbf{R}^n$ will be denoted by $\omega(x, \xi_1, \dots, \xi_l)$.

Iwaniec and Lutoborski prove the following result in [7]:

Lemma 1.2. *Let $D \in \mathbf{R}^n$ be a bounded convex domain. To each $y \in D$ there corresponds a linear operator $K_y : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ defined by*

$$(K_y\omega)(x; \xi_1, \dots, \xi_{l-1}) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \dots, \xi_{l-1})\, dt, \tag{1.8}$$

and the decomposition

$$\omega = d(K_y\omega) + K_y(d\omega)$$

holds at any point y in D.

A homotopy operator $T : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ is defined by averaging K_y over all points y in D:

$$T\omega = \int_D \varphi(y) K_y\omega\, dy, \tag{1.9}$$

where $\varphi \in C_0^\infty(D)$ is normalized so that $\int \varphi(y)\, dy = 1$. Then by (1.9) we have

$$\omega = d(T\omega) + T(d\omega), \tag{1.10}$$

and Eq. (1.10) is extended to all differential forms $\omega \in L^s(D, \wedge^l)$ with $d\omega \in L^s(D, \wedge^{l+1})$.

The following definition belongs to Ding [4].

Definition 1.3. We say that the weight $w(x)$ satisfies the $A_r^\lambda(\Omega)$ condition, $r > 1$, $\lambda > 0$, write $w \in A_r^\lambda(\Omega)$, if $w(x) > 0$ a.e. and

$$\sup_B \left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\lambda(r-1)} < \infty$$

for any ball $B \subset \mathbf{R}^n$.

As $\lambda = 1$, the properties of $A_r(\Omega)$ are in [5]. Another kind of weight functions and its properties can be found in [3]. The following lemma is from Nolder [8].

Lemma 1.4. *Each Ω has a modified Whitney cover of cubes $W = \{Q_i\}$ which satisfy*

$$\bigcup Q_i = \Omega, \qquad \sum_{Q\in W} \chi_{\sqrt{5/4}Q} \leqslant N\chi_\Omega, \tag{1.11}$$

for all $x \in R^n$ and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$ then there exists a cube R ($\notin W$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$.

We also need the following generalized Hölder inequality.

Lemma 1.5. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then*

$$\|fg\|_{s,\Omega} \leqslant \|f\|_{\alpha,\Omega}\|g\|_{\beta,\Omega}$$

for any $\Omega \in \mathbf{R}^n$.

The following version of the weak reverse Hölder inequality appears in [8].

Lemma 1.6. *Let u be a differential form satisfying the A-harmonic equation* (1.7) *in Ω, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leqslant C|B|^{(t-s)/st}\|u\|_{t,\sigma B} \tag{1.12}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

The following local Caccioppoli-type inequality is from [8].

Lemma 1.7. *Let $u \in D'(\Omega, \wedge^l)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbf{R}^n$, $l = 1, 2, \ldots, n$ and $\sigma > 1$. Let $1 < s < \infty$. Then there exists a constant C, independent of u, such that*

$$\|du\|_{s,B} \leqslant C\,\mathrm{diam}(B)^{-1}\|u - c\|_{s,\sigma B} \tag{1.13}$$

for all balls B with $\sigma B \subset \Omega$ and all closed forms c.

We extend Theorems A and B to A-harmonic tensors with $A_r^\lambda(\Omega)$ weight both locally and globally on a bounded domain $\Omega \subset \mathbf{R}^n$. We list our main results in Section 2 and give the proofs in Section 3. In Section 4 we show an application of the main results.

2. Main results

Theorem 2.1 is an extension of Theorem B with inequalities (1.3) and (1.5).

Theorem 2.1. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$, $1 < s < \infty$, be an A-harmonic tensor in a bounded domain $\Omega \subset R^n$ and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be a homotopy operator defined in* (1.9). *Assume that $\rho > 1$, and $w \in A_r^\lambda(\Omega)$ for some $r > 1$ and $\lambda > 0$. Then, for any ball B such that $\rho B \subset \Omega$, there exists a constant C, independent of u, such that*

$$\left(\int_B |Tu|^s w^\alpha \, dx\right)^{1/s} \leqslant C|B| \operatorname{diam}(B) \left(\int_{\rho B} |u|^s w^{\lambda\alpha} \, dx\right)^{1/s} \tag{2.1}$$

and

$$\left(\int_B \left|\nabla(Tu)\right|^s w^\alpha \, dx\right)^{1/s} \leqslant C|B| \left(\int_{\rho B} |u|^s w^{\lambda\alpha} \, dx\right)^{1/s} \tag{2.2}$$

for any real number α with $0 < \alpha < 1$.

The following theorem is an extension of Theorem B with inequality (1.4).

Theorem 2.2. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be an A-harmonic tensor in a bounded domain $\Omega \subset \mathbf{R}^n$ such that $du \in L^s_{\mathrm{loc}}(\Omega, \wedge^{l+1})$ and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be a homotopy operator defined in* (1.9). *Assume that $\sigma > 1$, and $w \in A_r^\lambda(\Omega)$ for some $r > 1$ and $\lambda > 0$. Then, for any ball B such that $\sigma B \subset \Omega$, we have*

$$\left(\int_B |dTu|^s w^\alpha \, dx\right)^{1/s} \leqslant C|B| \left(\int_{\sigma B} |u|^s w^{\lambda\alpha} \, dx\right)^{1/s} \tag{2.3}$$

for any real number α with $0 < \alpha < 1$.

Next theorem is about the global results of Theorem 2.1 and 2.2.

Theorem 2.3. *Let $u \in L^s(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$, $1 < s < \infty$, be an A-harmonic tensor in a bounded domain $\Omega \subset \mathbf{R}^n$ and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be a* *homotopy operator defined in* (1.9). *Assume that* $\rho > 1$, *and* $w \in A_r^\lambda(\Omega)$ *for some* $r > 1$ *and* $\lambda > 0$. *Then, there exists a constant* C, *independent of* u, *such that*

$$\left(\int_\Omega |Tu|^s w^\alpha \, dx\right)^{1/s} \leqslant C\left(\int_\Omega |u|^s w^{\lambda\alpha} \, dx\right)^{1/s} \tag{2.4}$$

and

$$\left(\int_\Omega \left|\nabla(Tu)\right|^s w^\alpha \, dx\right)^{1/s} \leqslant C\left(\int_\Omega |u|^s w^{\lambda\alpha} \, dx\right)^{1/s}, \tag{2.5}$$

also

$$\left(\int_\Omega \left|d(Tu)\right|^s w^\alpha \, dx\right)^{1/s} \leqslant C\left(\int_\Omega |u|^s w^{\lambda\alpha} \, dx\right)^{1/s} \tag{2.6}$$

for any real number α *with* $0 < \alpha < 1$.

One can get various versions of the imbedding inequalities when λ takes different values, for example, $\lambda = 1$, or $\lambda = 1/\alpha$.

3. Proofs of the main results

Proof of Theorem 2.1. Choose $t = s/(1-\alpha)$ for given $1 < s < \infty$ and any $0 < \alpha < 1$; then $1 < s < t$. Next, we choose m with $m = s/(\alpha\lambda(r-1)+1)$ for any given $r > 1$, apply Hölder inequality and (1.3) to obtain

$$\begin{aligned}
&\left(\int_B |Tu|^s w^\alpha \, dx\right)^{1/s} \\
&\quad = \left(\int_B \left(|Tu| w^{\alpha/s}\right)^s dx\right)^{1/s} \\
&\quad \leqslant \left(\int_B |Tu|^t \, dx\right)^{1/t} \left(\int_B (w^{\alpha/s})^{st/(t-s)} \, dx\right)^{(t-s)/(st)} \\
&\quad \leqslant C_1 |B| \operatorname{diam}(B) \|u\|_{t,B} \left(\int_B w^{\alpha t/(t-s)} \, dx\right)^{(t-s)/(st)}.
\end{aligned} \tag{3.1}$$

Use weak reverse Hölder inequality (1.12),

$$\|u\|_{t,B} \leqslant C_2 |B|^{(m-t)/mt} \|u\|_{m,\rho B} \tag{3.2}$$

and

$$\|u\|_{m,\rho B} = \left(\int_{\rho B} \left(|u| w^{\alpha\lambda/s} w^{-\alpha\lambda/s}\right)^m dx \right)^{1/m}$$
$$\leqslant \left(\int_{\rho B} |u|^s w^{\alpha\lambda} \, dx \right)^{1/s} \left(\int_{\rho B} (1/w)^{\alpha\lambda m/(s-m)} \, dx \right)^{(s-m)/sm}. \quad (3.3)$$

By the choice of m, $(\alpha\lambda m)/(s-m) = 1/(r-1)$ and $(m-t)/mt = -\alpha(1+\lambda(r-1))/s$. Since $w \in A_r^\lambda(\Omega)$, we have

$$\left(\int_B w^{\alpha t/(t-s)} \, dx \right)^{(t-s)/st} \left(\int_{\rho B} (1/w)^{\alpha\lambda m/(s-m)} \, dx \right)^{(s-m)/sm}$$
$$\leqslant |\rho B|^{\alpha/s + \alpha\lambda(r-1)/s}$$
$$\times \left(\frac{1}{|\rho B|} \int_{\rho B} w \, dx \right)^{\alpha/s} \left(\frac{1}{|\rho B|} \int_{\rho B} \left(\frac{1}{w}\right)^{1/(r-1)} dx \right)^{\alpha\lambda(r-1)/s}$$
$$\leqslant C_3 |\rho B|^{\alpha(1+\lambda(r-1))/s}. \quad (3.4)$$

Combining (3.1)–(3.4), we get

$$\left(\int_B |Tu|^s w^\alpha \, dx \right)^{1/s} \leqslant C_4 |B| \operatorname{diam}(B) \left(\int_{\rho B} \left(|u|^s w^{\alpha\lambda}\right) dx \right)^{1/s}.$$

The inequality (2.1) is proved.

The proof of inequality (2.2) is similar as the proof of (2.1). □

Proof of Theorem 2.2. Let $t = s/(1-\alpha)$. By Lemma 1.5,

$$\left(\int_B |dTu|^s w^\alpha \, dx \right)^{1/s}$$
$$= \left(\int_B \left(|dTu| w^{\alpha/s}\right)^s dx \right)^{1/s}$$
$$\leqslant \left(\int_B \left|d(Tu)\right|^t dx \right)^{1/t} \left(\int_B (w^{\alpha/s})^{st/(t-s)} \, dx \right)^{(t-s)/st}$$
$$= \left\|d(Tu)\right\|_{t,B} \left(\int_B w \, dx \right)^{\alpha/s}. \quad (3.5)$$

Using (1.4), (1.13) and (1.12), and choosing m such that $m = s/(\alpha\lambda(r-1)+1)$ for given $r > 1$, we have

$$\begin{aligned}\|d(Tu)\|_{t,B} &\leqslant \|u\|_{t,B} + C_1|B|\operatorname{diam}(B)\|du\|_{t,B}\\ &\leqslant \|u\|_{t,B} + C_2|B|\operatorname{diam}(B)\operatorname{diam}(B)^{-1}\|u\|_{t,B}\\ &= \bigl(1+C_2|B|\bigr)\|u\|_{t,B} \leqslant C_3|B|^{(m-t)/mt}\|u\|_{m,\sigma B}.\end{aligned}\tag{3.6}$$

Then using (3.3) and (3.4) with $\rho = \sigma$,

$$\left(\int_B |dTu|^s w^\alpha\,dx\right)^{1/s} \leqslant C|B|\left(\int_{\sigma B} |u|^s w^{\lambda\alpha}\,dx\right)^{1/s}$$

for any real number α with $0 < \alpha < 1$. □

We use Lemma 1.4 to prove Theorem 2.3 for the global imbedding inequalities on a bounded domain Ω.

Proof of Theorem 2.3. By Lemma 1.4, there exists a Whitney cover $W = \{Q_i\}$ of Ω. In particular, we can choose $1 < \rho \leqslant \sqrt{5/4}$ in Theorem 2.1, so that

$$\begin{aligned}\int_\Omega |Tu|^s w^\alpha\,dx &\leqslant \sum_{Q\in W}\int_Q |Tu|^s w^\alpha\,dx \leqslant \sum_{Q\in W} C_1|Q|\operatorname{diam} Q\int_{\rho Q}|u|^s w^{\alpha\lambda}\,dx\\ &\leqslant C_2|\Omega|\operatorname{diam}\Omega\sum_{Q\in W}\int_{\rho Q}|u|^s w^{\alpha\lambda}\,dx\,\chi_{\rho Q}\\ &\leqslant C_3\int_\Omega |u|^s w^{\alpha\lambda}\,dx\sum_{Q\in W}\chi_{\sqrt{5/4}Q}\\ &\leqslant C_4\int_\Omega |u|^s w^{\alpha\lambda}\,dx\,N\chi_\Omega \leqslant C_5\int_\Omega |u|^s w^{\alpha\lambda}\,dx.\end{aligned}$$

Thus, inequality (2.4) in Theorem 2.3 is proved.

The proofs of inequalities (2.5) and (2.6) are similar. □

4. Applications

As an application of Theorem 2.1, we prove the following theorem, which is the extension of Theorem A to $A_r^\lambda(\Omega)$ weighted version.

Theorem 4.1. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be an A-harmonic tensor in a bounded domain $\Omega \subset \mathbf{R}^n$ and $T : L^s(\Omega, \wedge^l) \to W^{1,s}(\Omega, \wedge^{l-1})$ be a* *homotopy operator defined in* (1.9). *Assume that* $\sigma > 1$, *and* $w \in A_r^\lambda(\Omega)$ *for some* $r > 1$ *and* $\lambda > 0$. *Then, there exists a constant* C, *independent of* u, *such that*

$$\|Tu\|_{W^{1,s}(B),w^\alpha} \leqslant C|B|\|u\|_{s,\sigma B,w^{\alpha\lambda}} \tag{4.1}$$

for all balls B *with* $\sigma B \subset \Omega$ *and any real number* α *with* $0 < \alpha < 1$.

Proof. By (1.6), (2.1) and (2.2) we have

$$\begin{aligned}\|Tu\|_{W^{1,s}(B),w^\alpha} &= (\operatorname{diam} B)^{-1}\|Tu\|_{s,B,w^\alpha} + \|\nabla(Tu)\|_{s,B,w^\alpha}\\ &\leqslant |B|(\operatorname{diam} B)^{-1}C_1 \operatorname{diam} B\|u\|_{s,\sigma B,w^{\alpha\lambda}}\\ &\quad + C_2|B|\|u\|_{s,\sigma B,w^{\alpha\lambda}} \leqslant C_3|B|\|u\|_{s,\sigma B,w^{\alpha\lambda}}.\end{aligned}$$

Next, use Theorem 4.1 we prove the following Sobolev–Poincaré imbedding inequality.

Theorem 4.2. *Let* $du \in L^s_{\mathrm{loc}}(\Omega, \wedge^{l+1})$, $l = 0, 1, 2, \ldots, n$, $1 < s < \infty$, *be an* A*-harmonic tensor in a bounded domain* $\Omega \subset \mathbf{R}^n$ *and* $T : L^s(\Omega, \wedge^l) \to W^{1,s}(\Omega, \wedge^{l-1})$ *be a homotopy operator defined in* (1.9). *Assume that* $\sigma > 1$, *and* $w \in A_r^\lambda(\Omega)$ *for some* $r > 1$ *and* $\lambda > 0$. *Then, there exists a constant* C, *independent of* u, *such that*

$$\|u - u_B\|_{W^{1,s}(B),w^\alpha} \leqslant C|B|\|du\|_{s,\sigma B,w^{\alpha\lambda}} \tag{4.2}$$

for all balls B *with* $\sigma B \subset \Omega$ *and any real number* α *with* $0 < \alpha < 1$.

Proof. Let $u_B = u - T(du)$. By Theorem 4.1 and the definition of T, we have

$$\|u - u_B\|_{W^{1,s}(B),w^\alpha} = \big\|T(du)\big\|_{W^{1,s}(B),w^\alpha} \leqslant C|B|\|du\|_{s,\sigma B,w^{\alpha\lambda}}.$$

References

[1] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Rational Mech. Anal. 63 (1977) 337–403.

[2] B. Bojarski, T. Iwaniec, Analytical foundations of the theory of quasiconformal mappings in $\mathbf{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983) 257–324.

[3] S. Ding, P. Shi, Weighted Poincaré-type inequalities for differential forms in $L^s(\mu)$-averaging domains, J. Math. Anal. Appl. 227 (1998) 200–215.

[4] S. Ding, Estimates of weighted integrals for differential forms, J. Math. Anal. Appl. 256 (2001) 312–323.

[5] J.B. Garnett, Bounded Analytic Functions, Academic Press, New York, 1970.

[6] J. Heinonen, T. Kilpelainen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, 1993.

[7] T. Iwaniec, A. Lutoborski, Integral estimates for null Lagrangians, Arch. Rational Mech. Anal. 125 (1993) 25–79.

[8] C.A. Nolder, Hardy–Littlewood theorems for A-harmonic tensors, Illinois J. Math. 43 (1999) 613–631.